General perspectives on modelling and problem solving

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mathematical modelling

Example: population growth

$$p = C \cdot e^{at}$$

parameters

Choosing the model: model selection

Choosing the parameters: parameter estimation

What can we do with the model?

1) Understand and explain:

$$p' = a\,p$$

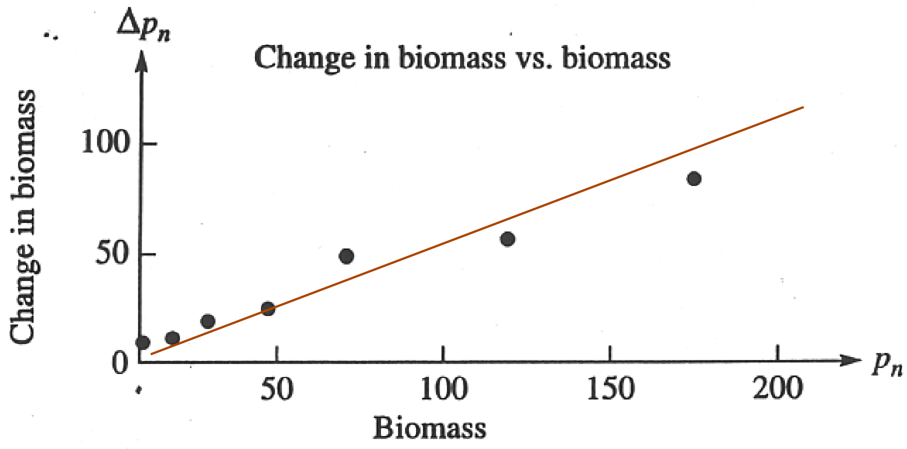
”the growth rate is proportional to the population”

2) Solve a problem: what is the population in 2020?

How find this model from data?

Figure 1.7
Growth of a yeast culture versus time in hours; data from R. Pearl, "The Growth of Population," *Quart. Rev. Biol.* 2(1927): 532–548

Time in hours n	Observed yeast biomass p_n	Change in biomass $p_{n+1} - p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	



Although the graph of the data does not lie precisely along a straight line passing exactly through the origin, it can be *approximated* by such a straight line. Placing a ruler over the data to approximate a straight line through the origin, we estimate the slope of the line to be about 0.5. Using the estimate $k = 0.5$ for the slope of the line, we hypothesize the proportionality model

$$\boxed{\Delta p_n = p_{n+1} - p_n = 0.5 p_n} \quad \Rightarrow \quad p' = 0.5/T\, p$$

yielding the prediction $p_{n+1} = 1.5 p_n$. This model predicts a population that increases forever, which is questionable. ■

Models and model types

Confused terminology: model, model type, problem, hypothesis

<u>Model</u>: a particular model i.e. a model *instance*

<u>Model type</u>: a class of models with similar mathematical properties

A critical decision: selecting the modelling approach!

Functions and equations

- linear
- quadratic
- nonlinear
- …

Dynamic models ≈ differential equations

- ordinary/partial
- linear/nonlinear
- …

Optimization models

- unconstrained/constrained
- linear programming
- nonlinear programming
- integer programming
- …

Each weekly module focuses on a particular main model type!

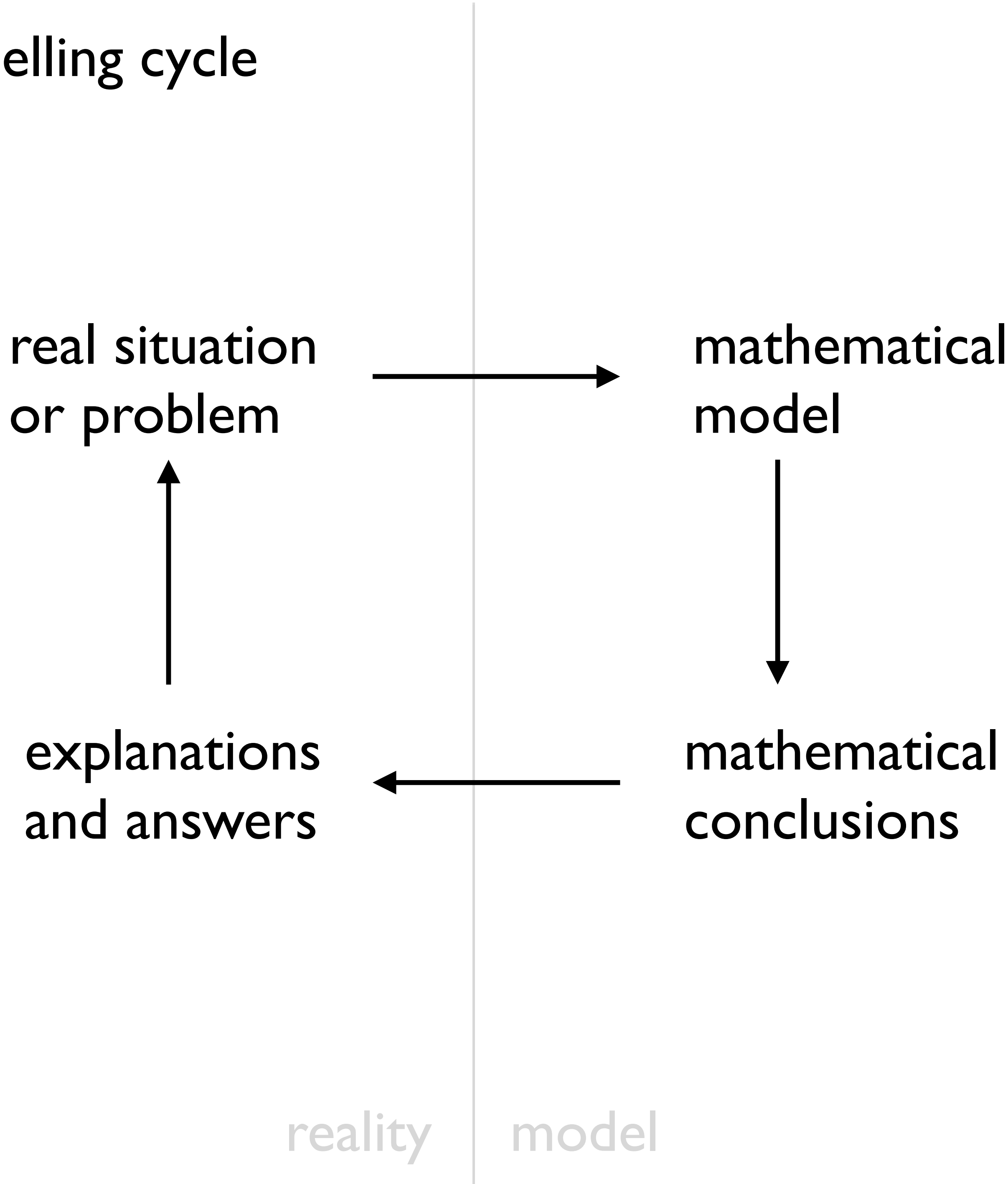
Standard application models

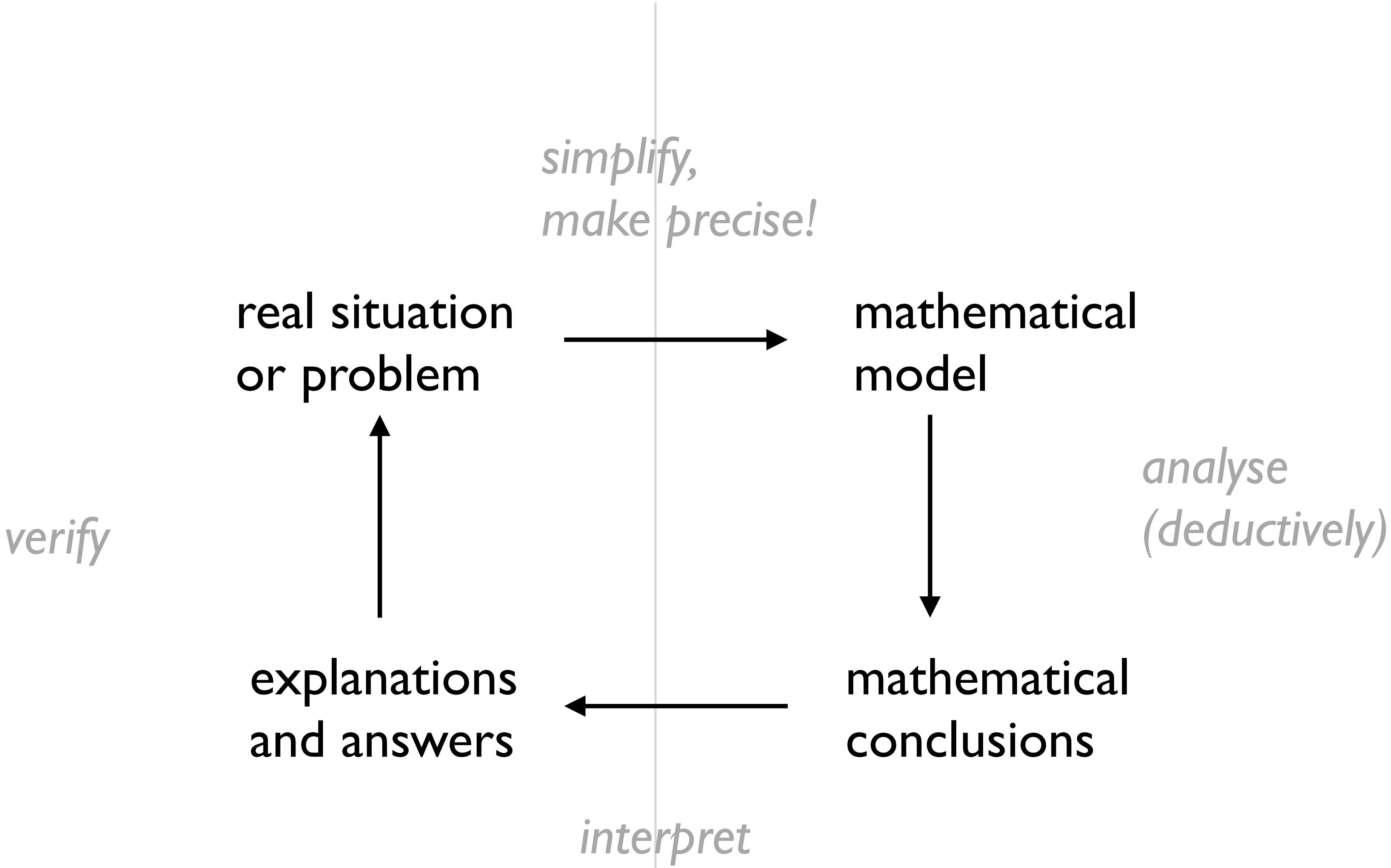
- exponential population growth, …
- vehicle routing problem, …
- … (thousands more)

Not always so clear what is a model/
application model/model type:

- the shortest path problem

The modelling cycle





Different people describe this process in different ways!

Can we improve?

think!

look at the data!

mechanistic
modelling
(deduction)
(or mathematical simplicity!)

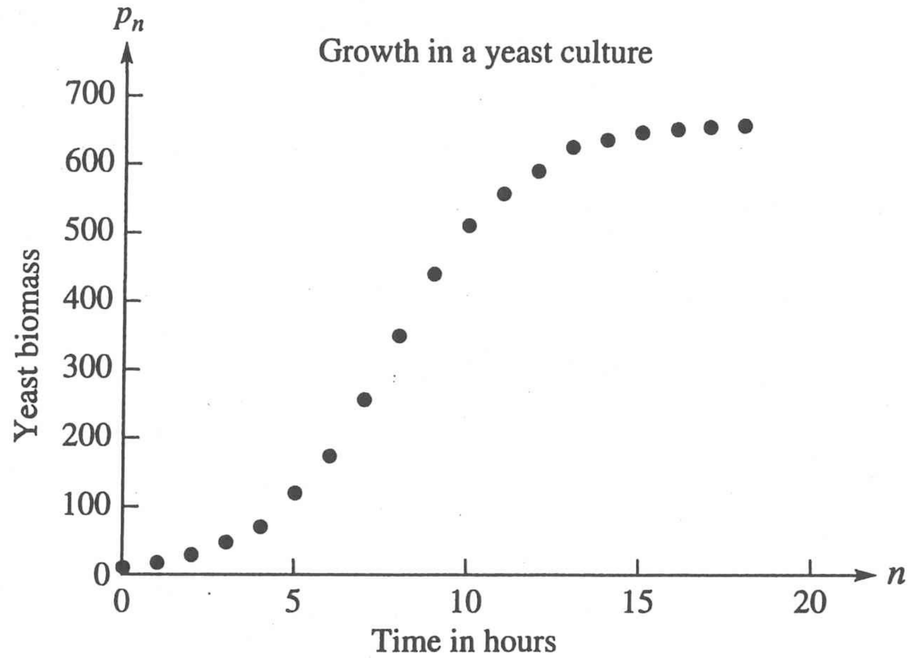
empirical
modelling
(induction)

Think…

- a maximum limit
- we probably want to keep exponential growth for low populations
- simple ways to do this?

Figure 1.8
Yeast biomass approaches a limiting population level

Time in hours n	Yeast biomass p_n	Change/hour $p_{n+1}-p_n$
0	9.6	8.7
1	18.3	10.7
2	29.0	18.2
3	47.2	23.9
4	71.1	48.0
5	119.1	55.5
6	174.6	82.7
7	257.3	93.4
8	350.7	90.3
9	441.0	72.3
10	513.3	46.4
11	559.7	35.1
12	594.8	34.6
13	629.4	11.4
14	640.8	10.3
15	651.1	4.8
16	655.9	3.7
17	659.6	2.2
18	661.8	



which causes the change Δp_n to become increasingly small as p_n approaches 665. Mathematically, this hypothesized model states that the change Δp_n is proportional to the product $(665-p_n)p_n$. To test the model, plot $(p_{n+1}-p_n)$ versus $(665-p_n)p_n$ to see if there is a reasonable proportionality. Then estimate the proportionality constant k.

Examining Figure 1.9, we see that the plot does reasonably approximate a straight line projected through the origin. We estimate the slope of the line approximating the data to be about $k \approx 0.00082$, which gives the model

$$p_{n+1} - p_n = 0.00082(665 - p_n)p_n \tag{1.2}$$

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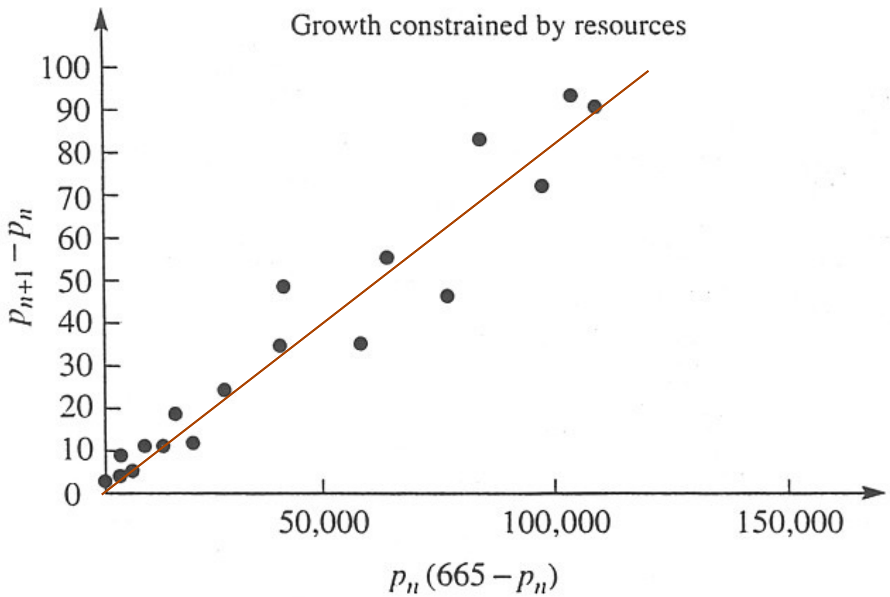
$p_{n+1} - p_n$	$p_n(665 - p_n)$
8.7	6291.84
10.7	11,834.61
18.2	18,444.00
23.9	29,160.16
48.0	42,226.29
55.5	65,016.69
82.7	85,623.84
93.4	104,901.21
90.3	110,225.01
72.3	98,784.00
46.4	77,867.61
35.1	58,936.41
34.6	41,754.96
11.4	22,406.64
10.3	15,507.36
4.8	9050.29
3.7	5968.69
2.2	3561.84



A refined model

A lot points towards the non-linear differential equation

$$p' = r\,(M - p)\,p$$

There is an analytical solution:

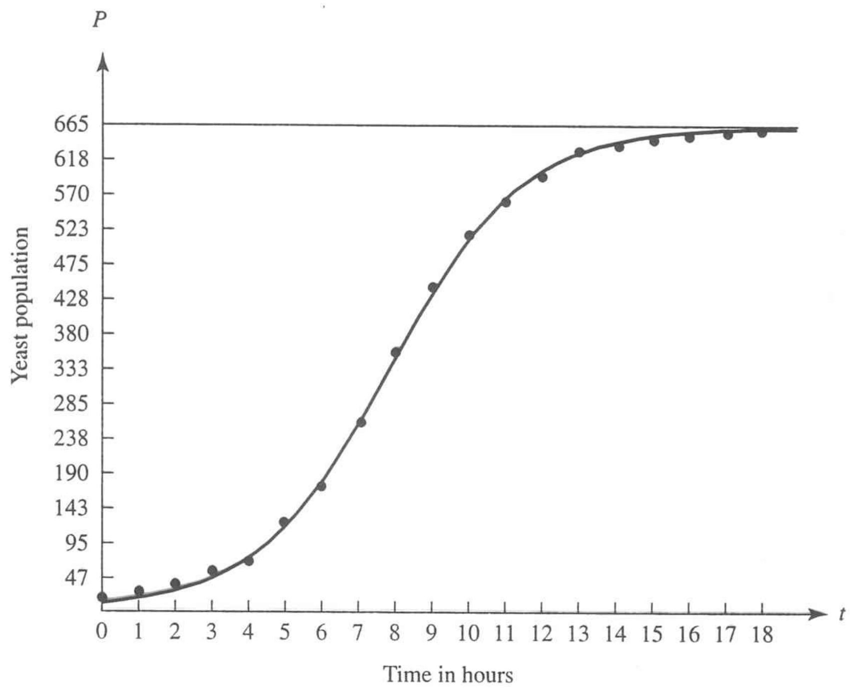
$$p(t) = \frac{a}{1 + c\,e^{-dt}}$$

known as
<u>logistic growth</u>

Room for more improvements: what about extinction?

Verification of the model

Figure 10.6
Logistic curve showing the growth of yeast in a culture based on the data from Table 10.1 and the Model (10.13); the small dots indicate the observed values



The simplicity and precision demanded by mathematics, as well as the iterative development process, often contributes to a successively improved understanding of the studied real system.

Table 10.2 Population of the United States from 1790 to 2000, with predictions from Equation (10.14)

Year	Observed population	Predicted population	Percent error
1790	3,929,000	3,929,000	0.0
1800	5,308,000	5,336,000	0.5
1810	7,240,000	7,227,000	−0.2
1820	9,638,000	9,756,000	1.2
1830	12,866,000	13,108,000	1.9
1840	17,069,000	17,505,000	2.6
1850	23,192,000	23,191,000	−0.0
1860	31,443,000	30,410,000	−3.3
1870	38,558,000	39,370,000	2.1
1880	50,156,000	50,175,000	0.0
1890	62,948,000	62,767,000	−0.3
1900	75,995,000	76,867,000	1.1
1910	91,972,000	91,970,000	−0.0
1920	105,711,000	107,393,000	1.6
1930	122,755,000	122,396,000	−0.3
1940	131,669,000	136,317,000	3.5
1950	150,697,000	148,677,000	−1.3
1960	179,323,000	159,230,000	−11.2
1970	203,212,000	167,943,000	−17.4
1980	226,505,000	174,941,000	−22.8
1990	248,710,000	180,440,000	−27.5
2000	281,416,000	184,677,000	−34.4

A simple model helps us to see that something new happens here!

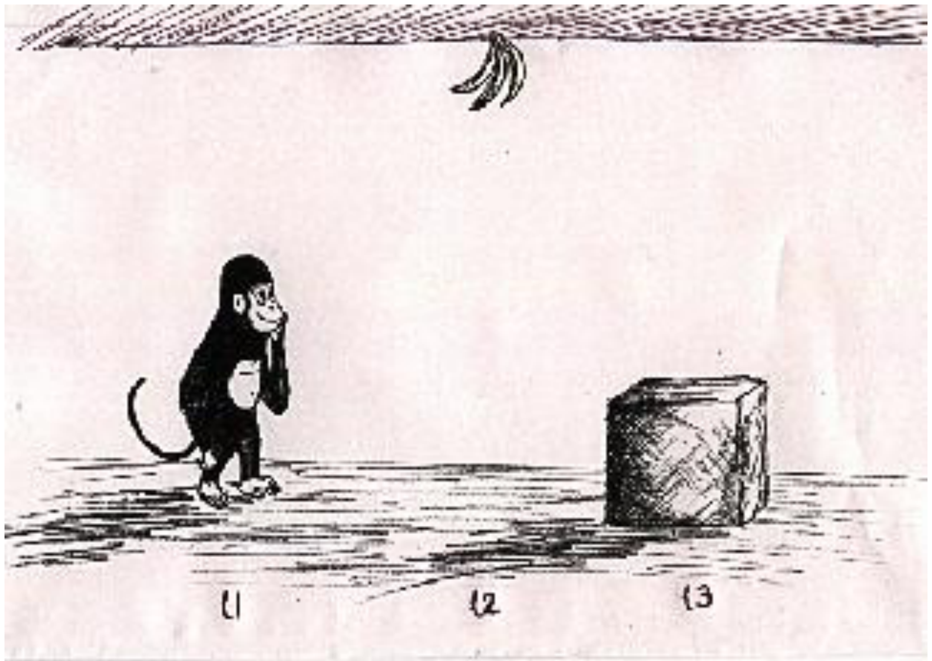
Modelling and problem solving

Modelling can be seen as *the first step of problem solving* for real problems.

It takes place before a precise mathematical model has been defined and therefore has an intuitive character.

problem solving

How can we make the most of our limited capacity?



Work in small manageable steps - (more power through many steps that you can actually implement)

Use effective tools - (more power in every step)

Use effective techniques - (more power by well chosen steps)

similarities with computing?

significant differences?

Since many steps are involved…

The notions of subproblems and subroutines become important.

We must be able to think also on a strategic level, and not just about the details.

We also need to *manage* our work and our time!

Typical workflow - easy or familiar problems

1. We easily understand the problem

2. We quickly see how to solve it

3. The problem is solved / implement the solution with no surprises

Typical workflow - intermediate problems

1. Understand the problem
2. Make a plan
3. Carry out the plan
4. Look back (check your result,
reflect on the process, ...)

(Polya)

Typical workflow – more difficult problems

1. Understand the problem
2. Make a plan
3. Carry out the plan
4. Look back

investigate for deeper understanding, define clearly

explore different approaches, begin with something simple!

when you fail you learn and go back

Continuously reflect, go back and revise, manage your time

This requires a lot of self-awareness!

You will have to struggle!

Never stop! Always do something!

We can always try to understand the problem better… Draw a figure, try small examples… Try to intuitively understand the problem and its solution…

Try to solve a simpler problem in a simple way…

Investigate extreme cases for easy thinking

If we expect our task to be a process in many steps, it becomes natural to engage in different subtasks that may shed light on the problem. Anything can be a clue!

investigate!

explore!

try things out -
different things

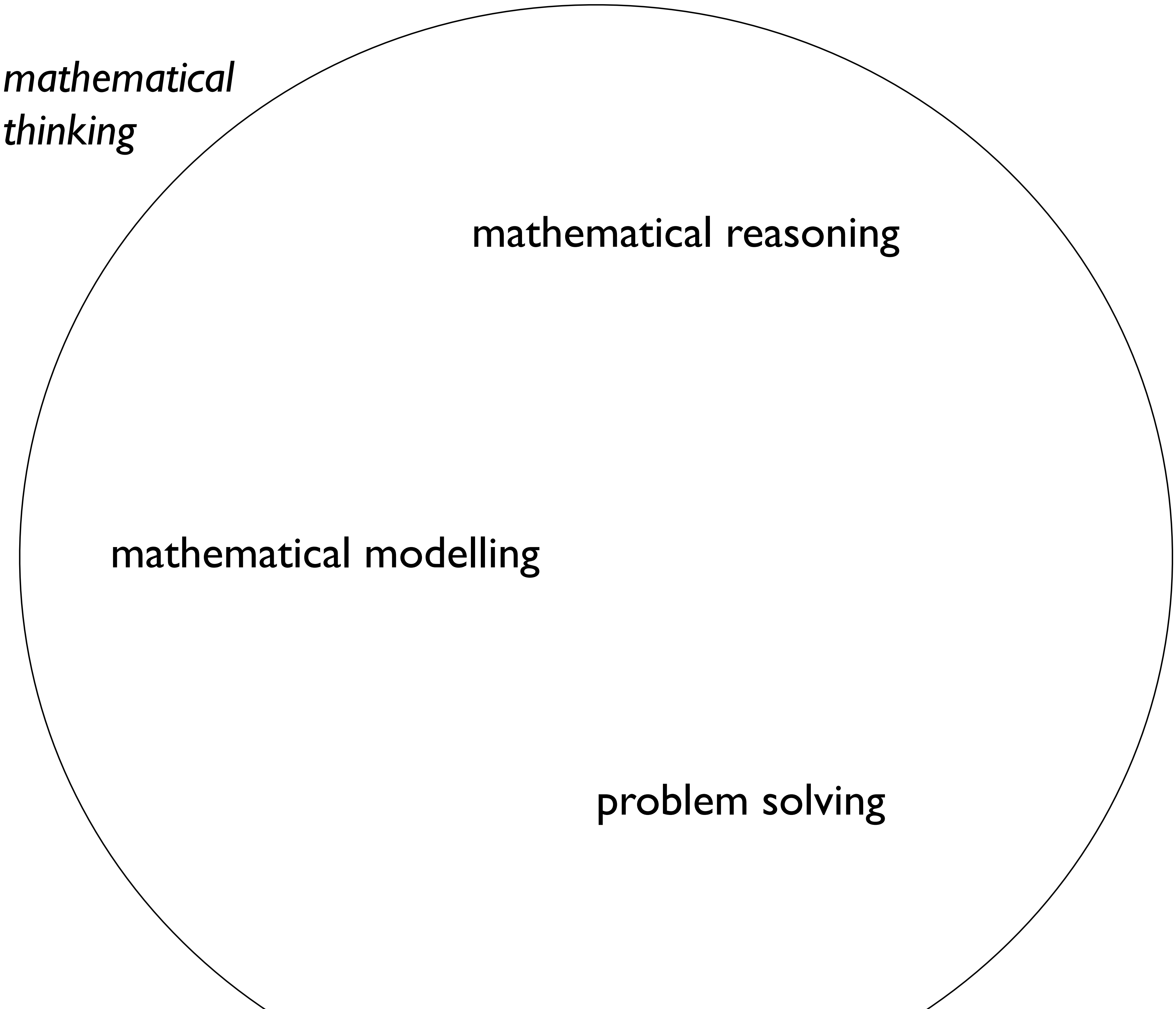
be careful!

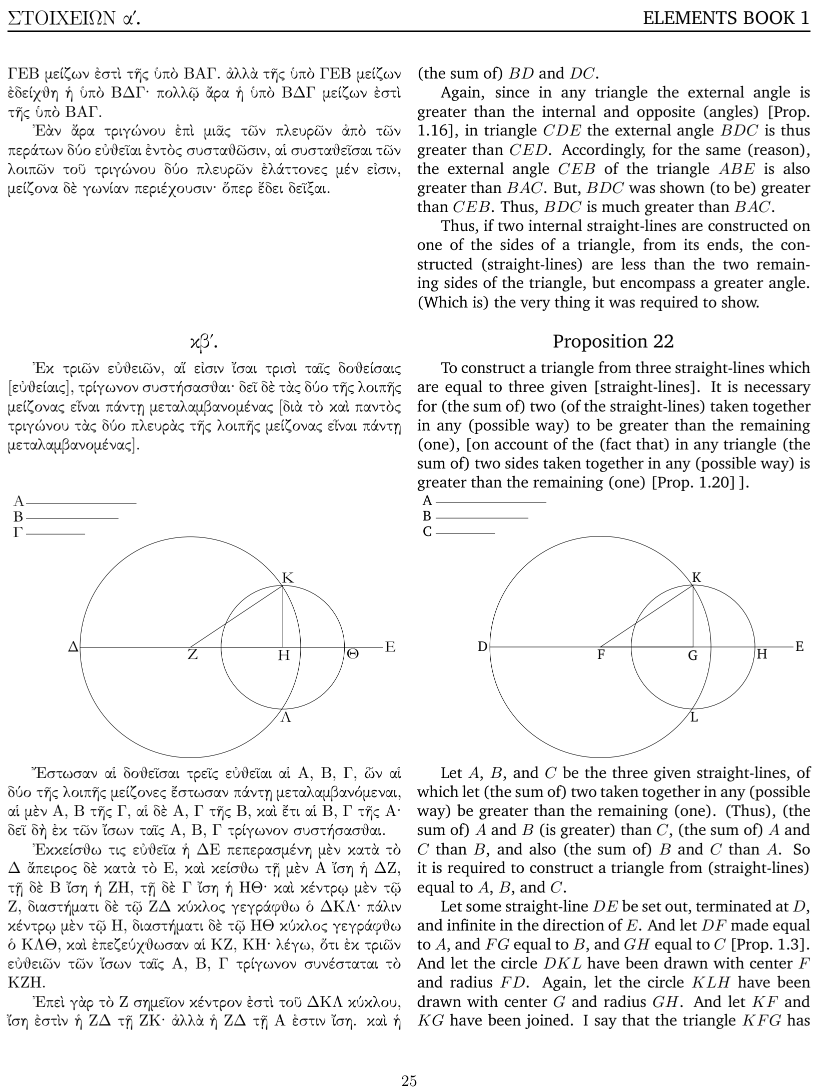
manage your
time

Dare to fail – you will find out. The tenth time you may get it right.

(anonymous student)

mathematical thinking


mathematical thinking
mathematical reasoning
mathematical modelling
problem solving



ΣΤΟΙΧΕΙΩΝ α′. ELEMENTS BOOK 1

ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ· πολλῷ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.

(the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

κβ′.

Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].

A B Γ Δ Z H Θ E K Λ

A B C D F G H E K L

Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ

Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KLH have been drawn with center G and radius GH. And let KF and KG have been joined. I say that the triangle KFG has

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Thinking creates knowledge!

What is needed to solve a problem?

very different balance
for different problems

knowledge needed for solving a problem = knowledge created by own thinking + knowledge from others

because of the variation
you often have to add
something here!

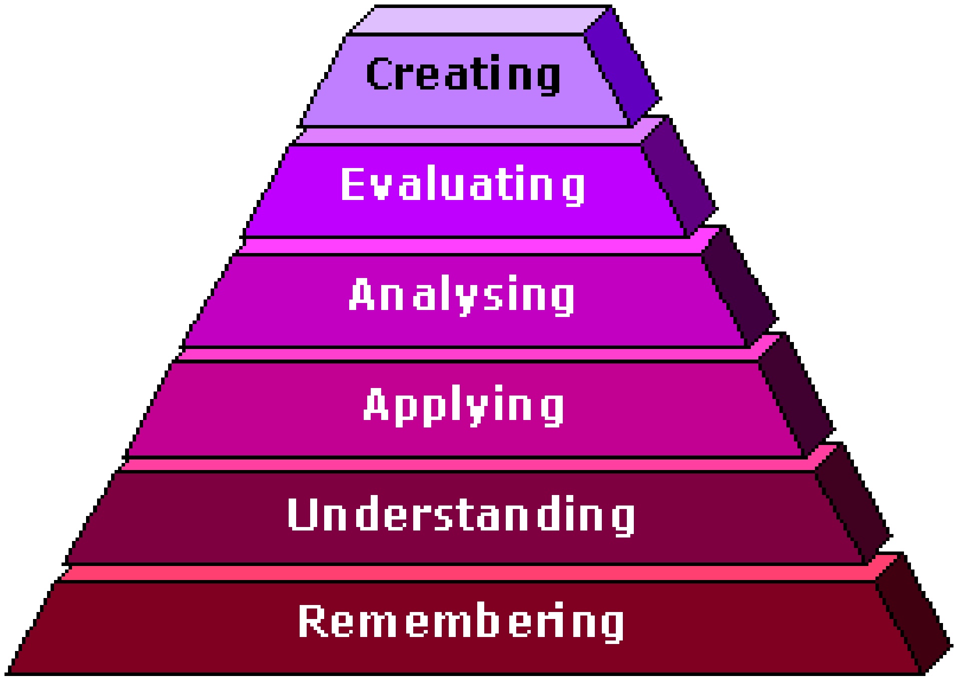
Many people think that mathematics must be explained in some particular format. This is wrong! It is the reasoning that must be sound.

A formula can be good - but sometimes a figure and a text is much easier.

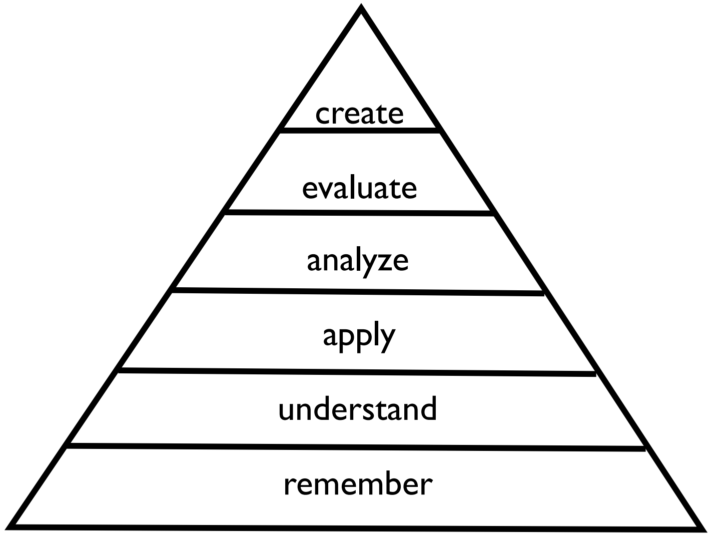
However, always take care so that statements are unambiguous!

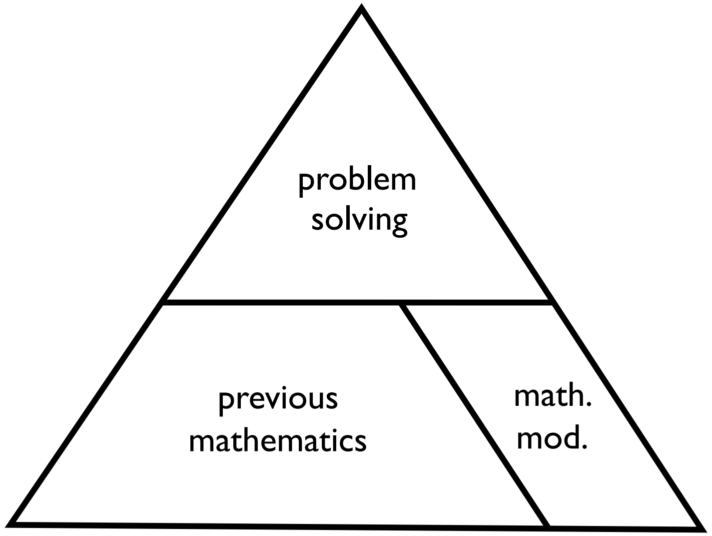
So explain as you like!

Bloom’s taxonomy (1956, improved by Anderson et al 2001)



Relation between Bloom's taxonomy and this course





The course adds two important pieces which complement the mathematics you already know!

Remember that we want a weekly meeting with all of you!
It can be very short if everything is fine.

END